Embroidering in Banach Space

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A Banach Space is a complete normed vector space.
A **Banach Space** is a complete normed vector space. We can

- measure **length** of a vector $\|x\|$. 
- compute **distance** between vectors $\|x - y\|$. 
- determine **convergence** of a Cauchy sequence of vectors.

Throughout this talk, $\mathcal{X}$ denotes a separable Banach space, endowed with the norm topology.
Some Preliminaries.

A **Banach Space** is a complete normed vector space.

**Examples:**

- All finite dimensional vector spaces.
- If $X$ is a compact Hausdorff space,

$$C(X) = \{ f : X \to \mathbb{C} : f \text{ is continuous} \}$$

equipped with the norm $\| f \| = \sup \{ |f(x)| : x \in X \}$

- The set

$$\ell^p = \left\{ (x_n) \in \mathbb{C}^\mathbb{N} : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}$$

equipped with the norm $\| (x_n) \|^p = \sum_{n=1}^{\infty} |x_n|^p$. 
A function $T : \mathcal{X} \rightarrow \mathcal{X}$ is a \textbf{linear operator} if

$$T(x + cy) = T(x) + cT(y)$$

for all $x, y \in \mathcal{X}$ and $x \in \mathbb{C}$.

We say that $T$ is \textbf{bounded} provided that

$$\sup\{\|Tx\| : \|x\| = 1\} < \infty$$
Let $S$ be a non-empty subset of $\mathcal{X}$, with closure denoted $\overline{S}$.

- If $S$ contains a non-empty open set, we say $S$ has interior.
- If $\overline{S}$ has interior, then $S$ is somewhere dense.
- If $\overline{S} = \mathcal{X}$, then $S$ is (everywhere) dense.
Suppose $\mathcal{M}$ is an infinite-dimensional closed subspace of $\mathcal{X}$.

**Definition**

An operator $T$ is said to be $\mathcal{M}$-hypercyclic if there exists some $x \in \mathcal{X}$ for which

$$M \cap \text{Orb}(T, x) = M \cap \{T^n(x) : n \geq 0\}$$

is dense in $\mathcal{M}$.

- When $T$ is $\mathcal{X}$-hypercyclic, we simply say $T$ is hypercyclic.
Theorem (Rolewicz 1969)
\[ \ell^2 \text{ supports a hypercyclic operator; } 2B \ (B \text{ is the backward shift}). \]
Existence.

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\( \ell^2 \) supports a hypercyclic operator; \( 2B \) (\( B \) is the backward shift).

Theorem (Ansari 1997, Bernal 1999)

Every infinite-dimensional Banach space supports a hypercyclic operator.
Existence.

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**Theorem (Ansari 1997, Bernal 1999)**

*Every infinite-dimensional Banach space supports a hypercyclic operator.*

**Theorem (Grivaux 2003)**

*If \( V \) is a countable, linearly independent dense subset of \( X \), then there exists an bounded operator \( T \) and a vector \( x \) so that \( \text{Orb}(T, x) = V \).*
### Existence.

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A motivating question.

Given a operator $T$, where can $\text{Orb}(T,x)$ be dense?
- Dense only in the unit ball?
- Dense only in a subspace?
- To what degree can I prescribe the closure of an operator’s orbit?
Theorem (Bourdon-Feldmen 2003)

Suppose that $\mathcal{X}$ is a Banach space, and that $T : \mathcal{X} \to \mathcal{X}$ a linear operator. If $\text{Orb}(T,x)$ is somewhere dense in $\mathcal{X}$, then $\text{Orb}(T,x)$ is dense in $\mathcal{X}$.
Question:

- If \( \text{Orb}(T,x) \cap M \) is somewhere dense in \( M \), then is \( \text{Orb}(T,x) \cap M \) dense in \( M \)?
Are somewhere dense orbits in $M$ dense in $M$?

**Question:**
- If $\text{Orb}(T,x) \cap M$ is somewhere dense in $M$, then is $\text{Orb}(T,x) \cap M$ dense in $M$?

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**Theorem**

*Let $\mathcal{X}$ be a separable Banach space and $M$ an infinite dimensional. For each non-empty subset $U$, open relative to $M$, there exists a bounded operator $T$ with $\overline{M \cap \text{Orb}(T,x)} = \overline{U}$.***
The Strategy

Want to sew the orbit of $T$ through $M$ but “embroider” only $U$. 

We’ll construct a countable, dense, linearly independent subset $V$ of $X$ that intersects $M$ only in a dense subset of $U$. 

Use Grivaux’s theorem to construct a bounded linear operator whose orbit is precisely $V$.

**Theorem (Grivaux’s Theorem)**

If $V$ is a countable linearly independent dense subset of $X$, then there exists an bounded operator $T$ and a vector $x$ so that $\text{Orb}(T,x) = V$. 

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The Strategy

Want to sew the orbit of $T$ through $M$ but “embroider” only $U$.

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2. Use Grivaux’s theorem to construct a bounded linear operator whose orbit is precisely $V$.

Theorem (Grivaux’s Theorem)

*If* $V$ *is a countable linearly independent dense subset of* $\mathcal{X}$, *then there exists an bounded operator* $T$ *and a vector* $x$ *so that* $\text{Orb}(T, x) = V$. 
Proof. Assume that $X$ is a separable Banach space. Then the norm topology is second countable.

- Let $\{B_n : n \in \mathbb{N}\}$ be a countable base for the norm topology.
- Let $\{U_n : n \in \mathbb{N}\}$ be a countable base for the subspace topology induced by $U$. 
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Let \( \{ U_n : n \in \mathbb{N} \} \) be a countable base for the subspace topology induced by \( U \).

Pick nonzero \( x_1 \in U_1 \). Then

- \( \text{span}(x_1) \) and \( M \) have empty interior in \( \mathcal{X} \).
- Let \( \{ B_n : n \in \mathbb{N} \} \) be a countable base for the norm topology.
- Let \( \{ U_n : n \in \mathbb{N} \} \) be a countable base for the subspace topology induced by \( U \).

Pick nonzero \( x_1 \in U_1 \). Then
- \( \text{span}(x_1) \) and \( M \) have empty interior in \( \mathcal{X} \).
- Hence \( \text{span}(x_1) \cup M \) has empty interior, so we may pick

\[
y_1 \in B_1 \setminus (\text{span}(x_1) \cup M)
\]
Suppose $n > 1$ and for all $k < n$, that $x_1, y_1, \ldots, x_k, y_k$ have been chosen so that

(i) $x_k \in U_k$ and $y_k \in B_k \setminus M$.

(ii) $\{x_i, y_i : 1 \leq i \leq k\}$ is linearly independent.

Since $\text{span}\{x_i, y_i : 1 \leq i \leq n\}$ is finite dimensional, it has empty interior in $M$, so we may pick

$$x_n \in U_n \setminus \text{span}\{x_i, y_i : 1 \leq i \leq n\}$$
Suppose $n > 1$ and for all $k < n$, that $x_1, y_1, \ldots, x_k, y_k$ have been chosen so that

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Since $\text{span}\{x_i, y_i : 1 \leq i \leq n\}$ is finite dimensional, it has empty interior in $\mathcal{M}$, so we may pick

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Since $\text{span}\{x_i, y_j : 1 \leq j < i \leq n\} \cup \mathcal{M}$ has empty interior in $\mathcal{X}$, we may pick

$$y_n \in B_n \setminus (\text{span}\{x_i, y_j : 1 \leq j < i \leq n\} \cup \mathcal{M})$$
By induction, this furnishes a countable linearly independent set dense $V$ in $\mathcal{X}$ so that

$$V \cap M = \{x_i : i \in \mathbb{N}\}$$

whose closure is $U$. By Grivaux’s Theorem, there exists an operator $T : \mathcal{X} \to \mathcal{X}$ and a vector $x$ with $\text{Orb}(T, x) = V$. But then $\text{Orb}(T, x) \cap M = \{x_i : i \in \mathbb{N}\}$, and so

$$\overline{\text{Orb}(T, x) \cap M} = \overline{\{x_i : i \in \mathbb{N}\}} = \overline{U}$$

as desired.
Embroidering in (in an infinite-dimensional closed subspace of an infinite dimensional) **Banach Space** (with the orbit of a bounded linear operator).
Embroidering in Banach Space
Embroidering in locally convex topological vector spaces.
Open Questions

- If $T$ is $\mathcal{X}$-hypercyclic, does there exists a proper nontrivial closed subspace $\mathcal{M}$ for which $T$ is $\mathcal{M}$-hypercyclic?
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- Can one classify all the proper nontrivial subspaces of $\mathcal{X}$ for which an operator is $\mathcal{M}$-hypercyclic for (if any)?
Open Questions

- If $T$ is $X$-hypercyclic, does there exist a proper nontrivial closed subspace $M$ for which $T$ is $M$-hypercyclic?
- Can one classify all the proper nontrivial subspaces of $X$ for which an operator is $M$-hypercyclic for (if any)?
- The Invariant Subspace Problem (ISP) for Hilbert Space and...
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- The Invariant Subspace Problem (ISP) for Hilbert Space and
- Yellow Complete Normed Vector Spaces (Bananach Spaces)
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- **Viewers like you!**

Thanks!