Mathematics in Economics

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UCLA
71 Nobel Laureates in Economics; 22 are mathematicians.
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- Last 20 years; 15/40
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By 2031, 100% of all Nobel Laureates in Economics in history will be mathematicians.
Game Theory and Mechanism Design

- **Game Theory**
  - Kenneth Arrow (Nobel Prize 1972)
  - Gérard Debreu (Nobel Prize 1983)
  - John Nash (Nobel Prize 1994)
  - John Harsanyi (Nobel Prize 1994)
  - Richard Selten (Nobel Prize 1994)

- **Mechanism Design**
  - Leonid Hurwicz (Nobel Prize 2007)
  - Eric Maskin (Nobel Prize 2007)
  - Roger Myerson (Nobel Prize 2007)
  - Alvin Roth (Nobel Prize 2012)
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John von Neumann

- Rigorously defined the notions of a game, a strategy, and an equilibrium.
A Brief History of Game Theory

John von Neumann

- Rigorously defined the notions of a game, a strategy, and an equilibrium.
- 1928, proved the minimax theorem;

Theorem

*For every two-person, zero-sum game with finitely many strategies, there exists a payoff $P$ and a mixed strategy for each player so that*

(a) *Given player 2’s strategy, the best payoff possible for player 1 is $V$, and*

(b) *Given player 1’s strategy, the best payoff possible for player 2 is $-V$.*

- “there could be no theory of games...without that theorem...I thought there was nothing worth publishing until the Minimax Theorem was proved.”
“Theory of Games and Economic Behavior” (1944) ignites the widespread study of game theory.

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- Albert Tucker begins advising students in game theory;
- David Gale (1949)
- John Nash (1950)
- Lloyd Shapley (1954)
A few definitions.

**Definition**

An $n$-dimensional simplex is a convex linear combination of $n + 1$ (affinely independent) points. Thus, for $\nu_1, \ldots, \nu_{n+1}$ we have

$$S = \left\{ \sum_{i=1}^{n+1} \alpha_i \nu_i : \alpha_i \geq 0, \sum_{i=1}^{n+1} \alpha_i = 1 \right\}$$

1. A 1-simplex is a line segment.
2. A 2-simplex is a triangle.
3. A 3-simplex is a tetrahedron.
Definition

- A \( k \)-face of an \( n \)-simplex is the \( k \)-simplex formed by the span of any subset of \( k + 1 \) vertices.
- We call an \( (n - 1) \)-face a facet.

1. The points \{A\}, \{B\}, \{C\} are 0-faces.
2. The line segments \(AB, BC, AC\) are 1-faces, or facets.
A simplicial subdivision of an $n$-simplex $S$ is a collection of (distinct) smaller $n$-simplices whose union is $S$, with the property that any two of them intersect in a common face, or not at all.

A simplicial subdivision of a 1-simplex is just a partition into smaller intervals.
A simplicial subdivision of a 2-simplex:
Definition

A Sperner Labeling or Coloring of a simplicial subdivision is a labeling of its vertices so that:

1. Each of the facets \( \{v_0, \ldots, v_n\} \) of the \( n \)-simplex receives exactly one number \( \{1, \ldots, n + 1\} \).

2. Each vertex in the subdivision is labeled by one of the facet numbers, so that none of the facets that lie on the \( j \)th facet receive the labeling \( j \).
Theorem (Sperner’s Lemma)

Any Sperner-labelled subdivision of an $n$-simplex must contain an odd number of fully labelled $n$-subsplexes.

When $n = 1$:
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When $n = 1$: 

2 \hspace{1cm} 1
Theorem (Sperner’s Lemma)

Any Sperner-labelled subdivision of an $n$-simplex must contain an odd number of fully labelled $n$-subsimplxes.

When $n = 1$: 

\[
\begin{array}{c}
 2 \quad \frac{1}{2} \quad 1 \\
\end{array}
\]
Theorem (Sperner’s Lemma)

Any Sperner-labelled subdivision of an $n$-simplex must contain an odd number of fully labelled $n$-subsimplxes.

When $n = 1$: 

![Diagram showing the subdivision of a line segment with labels 0, -1, 1, and 2.]
Theorem (Sperner’s Lemma)

Any Sperner-labelled subdivision of an n-simplex must contain an odd number of fully labelled n-subsimplexes.

When $n = 1$:
When \( n = 2 \).
When $n = 2$. 
When \( n = 2 \).
When $n = 2$. 
Theorem (Brouwer’s Fixed Point Theorem)

Any continuous function $f : D^2 \rightarrow D^2$ admits a fixed point, i.e. a point $x \in D^2$ so that $f(x) = x$.

Proof.

Suppose $f$ has no fixed points. Let $T$ the triangle with vertices $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, and $e_3 = (0, 0, 1)$. For each $n \in \mathbb{N}$ let $T^n$ be a triangulation of $T$ with mesh($T^n$) < $2^{-n}$. Write each $x \in T$ in Barycentric coordinates:

- $x = a_1 e_1 + a_2 e_2 + a_3 e_3$, with $f(x) = b_1 e_1 + b_2 e_3 + b_3 e_3$.
- $a_1 + a_2 + a_3 = b_1 + b_2 + b_3 = 1$. 

\[\square\]
Proof.

Define $\lambda(x) = \min\{i : b_i < a_i\}$ on the vertices of $T^n$. Since $a_i \neq b_i$ for all $i$, there must be some $i$ for which $b_i < a_i$ (since the coefficients have unit sum). It is easy to show that $\lambda(x)$ is a Sperner labeling.

1. For each $T^n$ there is a completely labeled sub-simplex, call it $T^n$, with vertices $v^n_1, v^n_2, v^n_3$ labeled 1, 2, 3, respectively.

2. By compactness, each of these sequences admits a convergent subsequence, but their distance tends to zero and hence these subsequences (reindexed by $n$) admit the same limit, call it $v$.

3. Observe that $f(v^n_1)_1 < (v^n_1)_1$ by construction. So by continuity,

$$f(v)_1 = f(\lim_{n \to \infty} v_n)_1 = \lim_{n \to \infty} f(v^n_1)_1 \leq \lim_{n \to \infty} (v_1)_1^n = v_1$$
Proof.

The same holds true for all \( i \in \{1, 2, 3\} \). So

\[
v = v_1 e_1 + v_2 e_2 + v_3 e_3
\]

\[
f(v) = f(v)_1 e_1 + f(v)_2 e_2 + f(v)_3 e_3
\]

And \( f(v)_i \leq v_i \) for all \( i \in \{1, 2, 3\} \). Since

\[
v_1 + v_2 + v_3 = f(v)_1 + f(v)_2 + f(v)_3 = 1
\]

it follows that \( f(v) = v \).
**Definition**

A set-valued function $\Phi : X \to 2^X$ is said to be closed if whenever

1. $x_n \to x_0$ in $X$,
2. $y_n \in \Phi(x_n)$, and
3. $y_n \to y_0$

implies that $y_0 \in \Phi(x_0)$.

**Theorem (Kakutani)**

*Suppose $\Phi$ is an closed set-valued function of the triangle so that $\Phi(x)$ is closed and convex for each $x$ in the triangle. Then there exists an $x_0 \in S$ so that $x_0 \in \Phi(x_0)$.***
Proof.

Let $T^n$ be a sequence of triangulations whose mesh tend to zero.

- For each vertex $x^n$ of $T^n$, pick an arbitrary $y^n \in \Phi(x^n)$.
- Define $\phi_n : \Delta_2 \to \Delta_2$ with $\phi_n(x^n) = y^n$. Extend by linearity (over barycentric coordinates) inside each sub-simplex [continuous by pasting lemma].

$$\lambda_1 x^1 + \lambda_2 x^2 + \lambda_3 x^3 \mapsto \lambda_1 y^1 + \lambda_2 y^2 + \lambda_3 y^3$$

- By Brouwer’s fixed point theorem, for each $n$ there is $z^n$ so that $\phi_n(z^n) = z^n$. 

\[\square\]
Proof.

$\{z^n\}_{n=1}^\infty$ is the sequence of fixed points for $\{\phi_n : \Delta_2 \rightarrow \Delta_2\}_{n=1}^\infty$.

- By compactness, assume that $z^n$ converges to $z \in \Delta_2$.
- $\mathcal{T}^n$ a subsimplex in $\mathcal{T}$ that contains $z^n$, with vertices $z_1^n, z_2^n, z_3^n$.

\[
z^n = \lambda_1 z_1^n + \lambda_2 z_2^n + \lambda_3 z_3^n
\]

\[
\phi_n(z^n) = \lambda_1^n y_1^n + \lambda_2^n y_2^n + \lambda_3^n y_3^n
\]
Proof.

Since \([0, 1]\) is compact, and \(y^n_i \in \Delta_2\), we can assume that

1. \(\lambda^n_i \to \lambda_i\) and \(\lambda_1 + \lambda_2 + \lambda_3 = 1\).
2. \(y^n_i \to y_i\).

and so

\[ z = \lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3 \]
Proof.

Since the mesh of $T^n$ tends to zero:

1. $z_i^n \to z$ (by compactness, possible dropping to a subsequence)
2. $y_i^n \in \Phi(z_i^n)$ and $y_i^n$ converges to some $y_i$.
3. By closedness, we have that $y_i \in \Phi(z)$.

Since $\Phi(z)$ is convex, this proves that

$$\lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3 = z \in \Phi(z)$$
Nash Equilibria

**Definition**

In an $n$-person game, a Nash equilibrium is a collection of strategies for which if each player is assumed to know the equilibrium strategies of the other players, then no player has anything to gain by changing only his own strategy unilaterally.

E.g. Rock, Paper, Scissors.

- Finite number of pure strategies: $\{Rock, \ Paper, \ Scissors\}$
- A mixed strategy is a probability distribution over one’s pure strategies: play Rock 50%, Paper 25%, Scissors 25%.
- A player’s collection of mixed strategies lives in $\Delta_2$, the triangle.
Let $\Delta_2^B$ denote Barack’s set of possible mixed strategies.

Let $\Delta_2^M$ denote Michelle’s set of possible mixed strategies.

Define a function $r_B : \Delta_2^M \to \Delta_2^B$ as Barack’s reaction correspondence; if Michelle plays $x \in \Delta_2^1$, then $r(x)$ are the set of mixed strategies that maximize my payoff. Define $r_M$ similarly.

Finally, let $r : \Delta_2^M \times \Delta_2^B \to \Delta_2^B \times \Delta_2^M$ via $r(x, y) = r(x) \times r(y)$. 
Note that $r(x)$ is non-empty, since the expected payoff is continuous which attains a maximum on a compact set.

Moreover, if $a, b \in r(x)$ are two dominating mixed strategies, then if I play $a$ with percentage $p$ and $b$ with percentage $(1 - p)$ then this still dominates $x$. So $r(x)$ is convex.

If $P_n$ and $Q_n$ are sequences of mixed strategies for which $P_n \to P$ and $Q_n \to Q$ and each $Q_n$ counters $P_n$, then $Q$ counters $P$. 
Hence $r : \Delta^M_2 \times \Delta^B_2 \rightarrow \Delta^B_2 \times \Delta^M_2$ has a fixed point. This fixed strategy is one in which neither of us can do better off if we deviate from our strategies unilaterally. Hence this mixed game has a Nash equilibrium.
Under reasonable assumptions, determination of Nash Equilibria allows one to predict the outcome of strategic and rational players.
A natural question.

- Under reasonable assumptions, determination of Nash Equilibria allows one to predict the outcome of strategic and rational players.
- Can we reverse engineer this?
Under reasonable assumptions, determination of Nash Equilibria allows one to predict the outcome of strategic and rational players.

Can we reverse engineer this?

Can we tailor the rules of a game to force desirable equilibria?
Under reasonable assumptions, determination of Nash Equilibria allows one to predict the outcome of strategic and rational players.

Can we reverse engineer this?

Can we tailor the rules of a game to force desirable equilibria?

If one can\textsuperscript{t} impose exogenously given moral standards on strategic individuals, then the next best thing is to make their best strategy one that adheres to them.
Mechanism Design

- Traditional game theory is concerned with studying and determining solution concepts and equilibria given a game.
Traditional game theory is concerned with studying and determining solution concepts and equilibria given a game.

Mechanism design, also known as “Reverse Game Theory” decides on a desirable outcome, and attempts to construct a game that influences player’s strategies to converge to that outcome.

Maskin, Meyerson, Hurwicz (2007)
In 1962, David Gale and Lloyd Shapley considered the following problem:

- Given \( n \) men and \( n \) women, all of which have strict preferences over one another:

\[
egin{align*}
\text{\( m_1 \)} &: \, w_1 \succ w_2 \succ w_3 \\
\text{\( m_2 \)} &: \, w_1 \succ w_3 \succ w_2 \\
\text{\( m_3 \)} &: \, w_2 \succ w_3 \succ w_1 \\
\text{\( w_1 \)} &: \, m_3 \succ m_2 \succ m_1 \\
\text{\( w_2 \)} &: \, m_2 \succ m_1 \succ m_3 \\
\text{\( w_3 \)} &: \, m_2 \succ m_3 \succ m_1
\end{align*}
\]
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\[
\begin{align*}
    m_1 : w_1 &> w_2 > w_3 & w_1 : m_3 > m_2 > m_1 \\
    m_2 : w_1 &> w_3 > w_2 & w_2 : m_2 > m_1 > m_3 \\
    m_3 : w_2 &> w_3 > w_1 & w_3 : m_2 > m_3 > m_1 
\end{align*}
\]

- Does there exist a divorce-proof matching?
Yes; there always exist divorce-proof, or “stable” matchings.

Proposal Algorithm (Gale, Shapley 1962)
Yes; there always exist divorce-proof, or “stable” matchings.

Single Ladies Algorithm (Knowles 2008):

If you liked it you should’ve put a ring on it.
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If you liked it you should’ve put a ring on it.

**Round 1:** Each man proposes to his first choice woman. Each woman selects exactly one man from her proposing suitors to keep in her “queue,” and rejects the rest.
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Round k: Each man rejected in round \( k - 1 \) proposes to his \( k^{th} \) choice woman. Each woman selects exactly one man from her proposing suitors and her current queue to place (or keep) in her queue, and rejects the rest.
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- Results in a stable matching.
Stable Marriages

\[ m_1 : w_1 \succ w_2 \succ w_3 \quad \text{and} \quad \text{w}_1 : m_3 \succ m_2 \succ m_1 \]
\[ m_2 : w_1 \succ w_3 \succ w_2 \quad \text{and} \quad \text{w}_2 : m_2 \succ m_1 \succ m_3 \]
\[ m_3 : w_2 \succ w_3 \succ w_1 \quad \text{and} \quad \text{w}_3 : m_2 \succ m_3 \succ m_1 \]

- **Round 1:**

  Round 1:

  Round 2:

  Round 3:
Stable Marriages

\[
\begin{align*}
m_1 : w_1 & \succ w_2 \succ w_3 & w_1 : m_3 & \succ m_2 \succ m_1 \\
m_2 : w_1 & \succ w_3 \succ w_2 & w_2 : m_2 & \succ m_1 \succ m_3 \\
m_3 : w_2 & \succ w_3 \succ w_1 & w_3 : m_2 & \succ m_3 \succ m_1
\end{align*}
\]

**Round 1:**
- \(m_1, m_2\) propose to \(w_1\); \(w_1\) rejects \(m_1\) and keeps \(m_2\).
Stable Marriages

\[ m_1 : w_1 \succ w_2 \succ w_3 \quad w_1 : m_3 \succ m_2 \succ m_1 \]
\[ m_2 : w_1 \succ w_3 \succ w_2 \quad w_2 : m_2 \succ m_1 \succ m_3 \]
\[ m_3 : w_2 \succ w_3 \succ w_1 \quad w_3 : m_2 \succ m_3 \succ m_1 \]

**Round 1:**
- \( m_1, m_2 \) propose to \( w_1 \); \( w_1 \) rejects \( m_1 \) and keeps \( m_2 \).
- \( m_3 \) proposes to \( w_2 \).
Stable Marriages

- \( m_1 : w_1 \succ w_2 \succ w_3 \)
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  - \( m_1, m_2 \) propose to \( w_1 \); \( w_1 \) rejects \( m_1 \) and keeps \( m_2 \).
  - \( m_3 \) proposes to \( w_2 \).

- **Round 2:**
Stable Marriages

\[ m_1 : w_1 \succ w_2 \succ w_3 \quad w_1 : m_3 \succ m_2 \succ m_1 \]
\[ m_2 : w_1 \succ w_3 \succ w_2 \quad w_2 : m_2 \succ m_1 \succ m_3 \]
\[ m_3 : w_2 \succ w_3 \succ w_1 \quad w_3 : m_2 \succ m_3 \succ m_1 \]

- **Round 1:**
  - \( m_1, m_2 \) propose to \( w_1 \); \( w_1 \) rejects \( m_1 \) and keeps \( m_2 \).
  - \( m_3 \) proposes to \( w_2 \).

- **Round 2:**
  - Disgruntled \( m_1 \) proposes to \( w_2 \); \( w_2 \) evicts \( m_3 \) and keeps \( m_1 \).
Stable Marriages

\[
\begin{align*}
  m_1 &: w_1 \succ w_2 \succ w_3 & w_1 &: m_3 \succ m_2 \succ m_1 \\
  m_2 &: w_1 \succ w_3 \succ w_2 & w_2 &: m_2 \succ m_1 \succ m_3 \\
  m_3 &: w_2 \succ w_3 \succ w_1 & w_3 &: m_2 \succ m_3 \succ m_1
\end{align*}
\]

- **Round 1:**
  - \(m_1, m_2\) propose to \(w_1\); \(w_1\) rejects \(m_1\) and keeps \(m_2\).
  - \(m_3\) proposes to \(w_2\).

- **Round 2:**
  - Disgruntled \(m_1\) proposes to \(w_2\); \(w_2\) evicts \(m_3\) and keeps \(m_1\).

- **Round 3:**
Stable Marriages

\[
m_1 : w_1 \succ w_2 \succ w_3 \quad w_1 : m_3 \succ m_2 \succ m_1 \\
m_2 : w_1 \succ w_3 \succ w_2 \quad w_2 : m_2 \succ m_1 \succ m_3 \\
m_3 : w_2 \succ w_3 \succ w_1 \quad w_3 : m_2 \succ m_3 \succ m_1
\]

- **Round 1:**
  - \(m_1, m_2\) propose to \(w_1\); \(w_1\) rejects \(m_1\) and keeps \(m_2\).
  - \(m_3\) proposes to \(w_2\).

- **Round 2:**
  - Disgruntled \(m_1\) proposes to \(w_2\); \(w_2\) evicts \(m_3\) and keeps \(m_1\).

- **Round 3:**
  - Heartbroken \(m_3\) proposes to \(w_3\); \(w_3\) accepts \(m_3\).
Stable Marriages

Round 1:
- $m_1, m_2$ propose to $w_1$; $w_1$ rejects $m_1$ and keeps $m_2$.
- $m_3$ proposes to $w_2$.

Round 2:
- Disgruntled $m_1$ proposes to $w_2$; $w_2$ evicts $m_3$ and keeps $m_1$.

Round 3:
- Heartbroken $m_3$ proposes to $w_3$; $w_3$ accepts $m_3$. 
Everyone gets married. Once a woman becomes engaged, she is always engaged to someone. So, at the end, there cannot be a man and a woman both unengaged, as he must have proposed to her at some point (since a man will eventually propose to everyone, if necessary) and, being unengaged, she would have to have said yes.

The marriages are stable. Let Alice be a woman and Bob be a man who are both engaged, but not to each other. If Bob prefers Alice to his current partner, he must have proposed to Alice before he proposed to his current partner. If Alice accepted his proposal, yet is not married to him at the end, she must have dumped him for someone she likes more, and therefore doesn’t like Bob more than her current partner. If Alice rejected his proposal, she was already with someone she liked more than Bob.
This is provably (Gale, Shapley) the best possible divorce-free matching.
• This is provably (Gale, Shapley) the best possible divorce-free matching
• But who the men and women were telling the truth about their preferences?
This is provably (Gale, Shapley) the best possible divorce-free matching.

But who the men and women were telling the truth about their preferences?

Could they have lied about their preferences to get a better mate?
Strategizing Suitors

- No stable matching algorithm exists for which stating the true preferences is a dominant strategy for every agent. (Roth 1982)
No stable matching algorithm exists for which stating the true preferences is a dominant strategy for every agent. (Roth 1982)

Provided the women are truthful, there is no individual or coalition strategy for which any individual or every member of the coalition benefits. (Dubins and Freedman 81, Roth 82).
In two-sided matching markets where one side has no realistic incentive to misrepresent their preferences:

1. National Medical Residency Matching Program
   Roth 84, 90, 91, 96, 97, 99, 00, 03, 03, 03, 10

2. Gastroenterologists, Orthopaedic surgeon fellowships
   Roth 03, 04, 05, 05, 06, 08, 09

3. Kidney Exchange
   Roth 03, 04, 05, 07, 06, 06, 07, 09, 09, 10, 10, 10, 11, 11, 11, 12, 12, 12

4. The School Choice Problem
   Roth 05, 09, 05, 06

5. Sorority Rush
   Roth 91

Roth innovated variants of the Gale-Shapley algorithm to design pareto-improved matching algorithms immune to strategic preference reporting.
Dr. Gizem Karaali and my 2010 REU Group at Claremont.
GSO for giving me the opportunity to speak.
Viewers like you!